

# CAUSAL SET DYNAMICS: A TOY MODEL

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## Abstract

We construct a quantum measure on the power set of non-cyclic oriented graphs of  $N$  points, drawing inspiration from 1-dimensional directed percolation. Quantum interference patterns lead to properties which do not appear to have any analogue in classical percolation. Most notably, instead of the single phase transition of classical percolation, the quantum model displays two distinct crossover points. Between these two points, spacetime questions such as: “does the network percolate?” have no definite or probabilistic answer.

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# 1 Introduction

The effort to formulate a discrete theory of Quantum Gravity has recently recovered some of its appeal, due to results from the quantization of general relativity [1], black hole thermodynamics [2] and string theory [3]. All suggest that the spectrum of excitations of a theory of quantum gravity must be discrete. It follows that counting is a natural method to define spacetime volume. A minimum framework for a discrete model of spacetime geometry brings in two key elements: number and order [4]. Number gives the local conformal factor or spacetime volume element. Causal order suffices to define light cones and this represents spacetime geometry up to a conformal factor. If we call “points” the objects that are being counted, and assume that the causal relations between points do not form closed timelike curves, then they provide us with a structure called a partially ordered set (poset).

A poset  $\mathcal{P}$  is a discrete set with a transitive acyclic relation, namely a relation  $\prec$  such that  $\forall x, y, z \in \mathcal{P}$ ,

$$x \prec y \text{ and } y \prec z \Rightarrow x \prec z, \quad (1.1)$$

$$x \prec y \text{ and } y \prec x \Rightarrow x = y. \quad (1.2)$$

For any two points  $x, y \in \mathcal{P}$ , the *Alexandrov set* [5] or *interval*  $[x, y]$  is defined by

$$[x, y] \stackrel{\text{def}}{=} \{z : x \prec z \prec y\}. \quad (1.3)$$

A partially ordered set is said to be *locally finite* if the Alexandrov sets are finite; it is then called a *Causal Set*. The axiom (1.2) ensures that a causal set has no closed timelike curves.

Several authors have proposed placing the causal set structure at the center of a discrete formulation of quantum gravity [6] [7]. Poset-generating models which have been considered range from quantum spin network models [8] to stochastic models inspired from percolation theory [9]. Sorkin and

collaborators have conjectured that the causal set structure alone may be sufficient to construct a quantum theory of spacetime [6]. The poset is a discrete approximation of a physical manifold, which reproduces some topological properties of the manifold being approximated that other models are not able to reproduce [10]. It has the structure of a topological space, with a topology defined by the causal order. A family of Posets of finite and increasing numbers of points, determines a family of projective finitary topological spaces whose inductive limit is the continuous manifold being approximated [11]. Moreover, the Poset constitutes a genuine “noncommutative” space from the point of view of a generalization of Gel’fand-Naimark theorem. In effect, to the Poset corresponds a “noncommutative”  $C^*$ -algebra of operator valued functions, which will be useful for constructing quantum physics on the Poset [10].

Independently of the particular mathematical construction that may give rise to one or another choice of partial ordering, it seems well worth the effort to find out what can be learned from the structure of Causal Sets *per se*, insofar as analyzing its potential to provide a discrete representation of spacetime geometry and the use that this may have in understanding various novel approaches to quantum gravity.

Progress towards what might be called a Causal Set representation of Quantum Gravity has been hindered by several factors, not least of which is the absence of a satisfactory dynamical formulation.

First of all, what does one mean by “dynamical formulation” , when the variables in question are the causal structure of spacetime itself? As often, one finds it helpful to first answer the analogous question in Classical Mechanics. A classical dynamical problem can be formulated as that of finding a projection operator from the set of all histories onto the subset of such histories that are solutions of the classical equations of motion. As long as there is a single classical history corresponding to any given initial data set, this formulation of the dynamical problem is equivalent to the conventional

one in terms of deterministic evolution equations. In quantum mechanics it is not very meaningful to consider a single history. Instead, one would like to recast the dynamical problem in terms of subsets of the set of histories, by asking whether a subset of histories, which is determined by particular properties, is more likely to be realised than its complement. One knows that a probability cannot be assigned to sets of histories, because interference leads to violations of the probability sum rules [12]. Nevertheless, a meaningful interpretation can be derived from a *quantum measure* on sets of histories [13], the quantum measure being a generalisation of the probabilistic measure which takes into account the possibility of interference. We will adopt this point of view here, summarising it briefly in section 2.

In the case of Causal Sets, the challenge is to find a dynamical formulation which might explain how causal sets with asymptotic properties resembling those of the spacetime we live in might come to be selected as being at least reasonably likely.

Markopoulou and Smolin have recently proposed a dynamical Causal Set model where spacelike slices are spin networks which connect to each other by means of null struts [8].

However this construction, as any local network-building algorithm, suffers from a lack of Lorentz invariance at least at small scales, due to its reliance on horizontal slices and a local scaffolding procedure. Whether or not effective Lorentz invariance can be recovered at large scales, one would like to avoid introducing a global rest frame at the Planck scale, where the foundations of the theory are being set.

What is meant by the term “Lorentz invariance” in the present context? This term refers to the amplitude function (or probability) on the set of posets, but it is only meaningful when considering the amplitude of posets that can be (approximately) embedded in Minkowsky space. For these posets, one may consider how the causal links would look in different reference frames. A link which in one reference frame looks to be purely timelike

and of small size, will in a highly boosted frame appear to be stretched out and almost null [4]. A Lorentz-invariant model should not privilege any one reference frame over another, so in any given frame one should observe both short links and elongated links. In contrast, a local lattice-building model which in a given frame is only allowed to connect nearby points on a regular lattice, would not be a Lorentz-invariant model.

Our purpose in this article is to propose a simple toy model with which we will derive a quantum measure on the set of Posets without introducing any a priori lattice structure. We also wish to explore what sort of questions such a dynamical causal set model should be able to answer.

In order to arrive at a model that is simple enough that computer computations can be performed on relatively large posets, we choose to set aside some of the other issues that have previously frustrated attempts to construct a realistic quantum measure model for Causal Set dynamics. In particular, the model which we present in this article introduces a labeling of the points by integers, and the amplitude is not required to be labeling invariant. We further simplify the problem by considering non-cyclic oriented graphs rather than posets, the difference being that transitive relations are relevant in a graph whereas they are not relevant in the partial ordering. Both invariances, labeling and transitivity, can be recovered in the end by summing over labelings and summing over all graphs that represent the same Poset.

In section 3 we will give the outline of our toy model and present the corresponding quantum measure. A method to derive computable expressions for the measure is then described in section 4, which will be applicable to sets of histories whose properties can be expressed by columns of the connectivity matrix. A few examples are evaluated numerically to reveal some of the structure of the model, including the measure of all histories with no black holes.

## 2 Quantum measure theory

Quantum mechanics can be described as a simple generalization of classical measure theory (or probability theory). A classical measure is a map from an algebra of “measurable sets” to the positive real numbers which satisfies

$$I_2(A, B) \equiv |A \sqcup B| - |A| - |B| = 0, \quad (2.1)$$

where  $\sqcup$  denotes the union of disjoint sets. The “no-interference” condition (2.1) permits a probabilistic interpretation for sets of histories in statistical mechanics.

In quantum mechanics, the quantity  $I_2(A, B)$  represents the interference term between the two sets of alternatives  $A, B$ , when interference occurs the condition (2.1) is violated and for that reason one cannot assign a probabilistic interpretation to the sum over histories formulation. Instead of (2.1), quantum theory respects a slightly weaker set of conditions, which defines a structure known as a “quantum measure”.

A quantum measure is positive real valued function which satisfies the conditions

$$|N| = 0 \Rightarrow |A \sqcup N| = |A|, \quad (2.2)$$

$$I_3(A, B, C) \equiv |A \sqcup B \sqcup C| - |A \sqcup B| - |A \sqcup C| - |B \sqcup C| + |A| + |B| + |C| = 0. \quad (2.3)$$

It is worth noting that the first axiom (2.2), which is not necessary in probability theory because it follows from (2.1), must be included as a separate axiom for the quantum measure because  $I_3 = 0$  in itself does not guarantee that sets with zero measure do not interfere with others.

Clearly, the axiomatic structure of any theory has much to say as to how a theory should be interpreted. Since the sum over histories formulation leads to a weaker structure than (2.1), one naturally expects that quantum theory will have a weaker predictive power than probability theory insofar as its ability to discern which histories are preferred by Nature. The precise nature

of this weaker predictive power, and the correct interpretation of the sum over histories formulation of quantum mechanics, are encoded in the structure of the axioms (2.2 - 2.3). Sorkin has shown that this structure sustains an interpretation based on so-called “preclusion rules”, which establish when it can be said that a certain set of histories is almost certain *not* to be realised in Nature. As one might expect from the form of the condition  $I_3(A, B, C) = 0$ , these preclusion rules invoke correlations between *three* events, pertaining to three disjoint regions of spacetime [13].

It is well worth stressing that our choice to use the quantum measure formalism rather than, say, canonical quantization, is forced upon us by the absence of any *a-priori* causal structure. Other pregeometrical theories, such as String Theory, may yet allow a canonical approach to a fundamental theory of Physics, by introducing a Newtonian time in an abstract world which generates our physical spacetime indirectly, perhaps through something like the Holographic Principle [14]. However, issues regarding black hole thermodynamics have been raised that would eventually have to be addressed [15]. In any event this approach is not available here: in the Causal Set formalism a canonical quantization would equate the abstract Newtonian time with the labeling of the points of the causal sets, thereby defeating the purpose of a pre-geometrical theory of Quantum Gravity.

In the remainder of this article we will limit ourselves to a yet weaker form of predictive statements than preclusion, which Sorkin refers to as “propensity”: When referring to a particular physical property, one partitions the space of histories in two disjoint subsets by distinguishing histories which do or do not have this property. If the measure of the set of histories which *do* have the property is much larger than its complement it can be said that it has a high propensity. The concept of propensity is useful when analysing the classical limit of a quantum theory. For example one might distinguish spacetimes that have black holes and those that do not; if the property of having one or more black holes has a very high propensity, this would constitute a

prediction of the theory in the classical limit.

### 3 A quantum measure model for directed non-cyclic graphs

#### 3.1 Posets, Causal sets and Directed Non-cyclic graphs

As mentioned in the introduction a poset  $\mathcal{P}$  is a discrete set with an antisymmetric transitive relation.

The transitivity rule (1.1) allows one to differentiate two types of relations: the *links*, which are relations that cannot be obtained from the transitive rule, and the *transitive* or *redundant* relations.

The causal structure does not depend on whether a particular relation is a link or a transitive relation, so in terms of pure gravity one can say that the two types of relation are physically equivalent. However there is a practical difference, which shows up when performing actual calculations or numerical simulations with causal sets. There is generally an enormous number of possible transitive routes between two points in a large Causal Set, so any algorithm which considers each possible route individually is only applicable to small causal sets, the limit being about 10 points. To our present knowledge, there is no generally applicable approximation scheme to do calculations with large causal sets.

The difficulty resides in the absence of a convenient (one-to-one) representation of posets. The most natural approach would be to represent a poset in terms of its “relations matrix”, where  $R_{ij} = 1$  if and only if  $x_j \preceq x_i$  and otherwise  $R_{ij} = 0$  ( $x_i, x_j \in \mathcal{P}; i, j \in \mathbb{N}$ ). The matrices  $\mathbf{R}$  satisfy the transitivity condition

$$R_{ij} = \theta\left(\sum_k R_{ik} R_{kj}\right), \quad (3.1)$$

where  $\theta(x) = 1$  if  $x > 0$  and zero otherwise. The computational complexity



of checking these equations for each of the  $N^2$  possible binary matrices grows like  $N^5$  (using the most straightforward algorithm). One could equally well choose to use the *link matrix*,  $\mathbf{L}$ , where  $L_{ij} = 1$  if and only if  $x_i \preceq x_j$  is a link, but of course this leads to the same computational problem. The limitation on the number of points with which one can work affects almost every poset-related calculation. For example, the counting of posets with a given number of points has only been solved up to  $N = 11$  [17]. For large values of  $N$  it has been shown to grow asymptotically like [16]

$$C \times 2^{\frac{N^2}{4} + \frac{3}{2}N} e^N N^{-(N+1)},$$

but the method that yields this result does not readily generalize to other poset calculations.

To avoid these problems, which originate from the transitivity condition, we will consider the set of *all* lower-triangular binary matrices, regardless of whether or not they include all possible transitive relations. We will refer to these matrices as *connectivity matrices*, and denote them by  $\mathbf{C}$ .

A connectivity matrix represents a directed non-cyclic graph, i.e. a set of points connected by arrows such that arrows do not form closed loops. Given such a graph, it is always possible to label the points with consecutive integers in such a way that arrows point from a lesser label to a greater one. One then arrives at a lower-triangular binary matrix, where each entry  $C_{ij}$  is equal to one if and only if there is an arrow in the graph from  $j$  to  $i$  (with  $j < i$ ), and zero otherwise. Some of the connections represented in the matrix  $\mathbf{C}$  may be links, while others will be transitive relations, but unlike the matrix  $\mathbf{R}$  it is not required that *all* transitive relations be represented as connections or arrows of the graph.

We will understand the connectivity matrix to represent a “history” or a possibility for “spacetime”. The sum over histories then takes the form of an unconstrained sum over binary arrays, which makes it relatively easier to apply standard analytical tools.

Of course in the end the purpose is not to compute the sum over all histories, but over specified subsets of histories, which satisfy one or another physical property of interest. A “physical property” is a property of the causal structure, i.e. one that does not pertain either to the labeling of the points or to transitive relations. The sum over graphs will then include the sum over all the causal orders with the given property.

There are interesting physical properties that condition the connectivity matrix without increasing the computational complexity to the same extent as condition (3.1). For such properties, a graph-based model can be expected to yield computable expressions. We will see some examples in section 4.

### 3.2 The “final question”

A quantum measure model can be constructed from two basic elements: an amplitude function on the set of histories, and a “final question”,  $Q_f$ , which by definition must refer only to the part of the histories in the causal future of the region of interest. The question  $Q_f$  must be well-posed, in the sense that each history,  $\gamma$ , in the Hilbert space,  $\mathcal{H}$ , should give one and only one answer. The answer set then gives a partition of the Hilbert space in a disjoint union of sets  $E_i$ , such that

$$\gamma \in E_i \Leftrightarrow Q_f(\gamma) = a_i, \quad (3.2)$$

where  $a_i, i = 1, \dots, n (n \geq 2)$ , represents an element of the answer set.

Given an amplitude  $a : \mathcal{H} \rightarrow \mathbb{C}$ , one can then construct the following function  $|\cdot|$  on the power set of  $\mathcal{H}$  (When the computational procedure at hand does not give a finite answer for *every* subset of histories, one requires that the quantum measure be well-defined on a sigma-algebra of “measurable sets”):

$$|A| = \sum_i \left| \sum_{\gamma \in A \cap E_i} a(\gamma) \right|^2. \quad (3.3)$$

One easily verifies that  $|\cdot|$  satisfies the axioms (2.2-2.3) of the quantum measure.

We have introduced a “final question” as part of the procedure to construct a quantum measure, but this question in itself is not of any particular interest. Eventually, one would like to be able to show that the dependence of the quantum measure on the final question vanishes asymptotically in the limit of very large posets. In other words, the final question would then be an artifice introduced for the sole purpose of performing the calculation. The aim of the quantum measure formalism is to address other questions, which one might call “physical” questions. These should be formulated in such a way that they refer to the history *before* the final conditions. Physical questions can be either about the “present” state of the system, with an appropriate definition of “present state”, or about the spacetime history. For each possible answer there is a set  $A_i$  of histories with answer  $a_i$ , and the relative values of the quantum measures for different possible answers will reveal what the model has to say regarding this question.

The quantum measure formalism can sometimes *contain* a canonically quantized model, in the following sense. One first defines a one-parameter family of questions, which are to form a complete set, in the sense that their combined answers describe a history completely. These questions can be stated as : “*what is the state of the system at time  $t$ ?*”. The answer set is given by the eigenstates of a complete set of commuting (configuration-space) observables. When the sets of histories corresponding to different answers  $E_j^t$  do not interfere, *and* the sum of the measures over all possible answers at time  $t$  is equal to one, then the quantum measure formalism will provide the same information as a canonical theory for that particular family of questions. This will occur when the unitarity condition  $I_2(E_j^t, E_k^t) = 2\delta_{jk}$

is satisfied, where

$$I_2(E_j^t, E_k^t) = \sum_i \left( \sum_{\gamma_1 \in E_j^t \cap E_i} \sum_{\gamma_2 \in E_k^t \cap E_i} (a(\gamma_1)a^*(\gamma_2) + a^*(\gamma_1)a(\gamma_2)) \right). \quad (3.4)$$

The choice of a particular one-parameter family of questions is analogous to a choice of “slicing” in canonical quantization. Other possible slicings, based on different choices of one-parameter families of questions, may well lead to different unitarity conditions. This sort of exercise of course is only relevant to the extent that one is interested in making contact with canonical quantization methods. Taking a pre-geometrical perspective, one might argue that other criteria should take precedence over unitarity in guiding the search of the correct quantum measure; if in the end it turned out that such criteria were to lead one to a unique quantum measure, then the reverse problem of finding the choice(s) of slicing for which a unitary canonical theory can be deduced would provide a satisfactory solution to the problem of time.

Here we will choose the following “final question”:

*“which of the points  $i < N$  emit an arrow towards  $N$ ?”*

By definition of the term “final question” we are assuming that the predictive power of this model is limited to points that do not lie to the future of  $N$ . We will label the points in such a way that  $N$  is the largest label and there are  $N - 1$  other points which may or may not be to the past of  $N$  but will certainly not be to its future.

The answer set of this question is then the set of binary words of length  $N - 1$ , where a bit is equal to 0 in the absence of an arrow and 1 denotes the presence of a connection. In terms of the connection matrix, the answer to the final question will be given by its last row,  $\vec{C}_N$ .

This final question generates a partition of the space of histories into disjoint subsets of connectivity matrices with a fixed lowest row.

There is a natural one-parameter family of questions associated to this particular choice of final question, namely those whose answers are the rows of the connectivity matrix. These are the questions: “which points emit arrows towards the point labeled by  $t$ ?”. Note that in this case the cardinality of the answer set grows like  $2^{t-1}$ .

### 3.3 The amplitude and the quantum measure

We will make the following ansatz for the amplitude (factorizability):

$$a(\mathbf{C}) \stackrel{\text{def}}{=} \prod_{m=2}^N a(\vec{C}_{m-1}, m-1 \rightarrow \vec{C}_m, m), \quad (3.5)$$

where

$$a(\vec{C}_{m-1}, m-1 \rightarrow \vec{C}_m, m) \stackrel{\text{def}}{=} A_{C_{mm-1}} \prod_{l < m-1} A_{C_{ml} C_{m-1l}}. \quad (3.6)$$

We will also require that the amplitude to create a connection  $\vec{C}_m$  be independent of the previous state,  $\vec{C}_{m-1}$ . Choosing

$$A_0 = A_{00} = iA_{01} = \sqrt{q}, \quad (3.7)$$

$$A_1 = A_{11} = iA_{10} = \sqrt{p} \quad (3.8)$$

and  $p + q = 1$  ( $p, q \in \mathbb{R}_{>0}$ ), we arrive at a quantum generalisation of a one-dimensional directed percolation model. In the one-dimensional directed percolation model,  $N$  points are labeled by consecutive integers as in our model, and each point can connect to any of the previous points with a constant probability  $p$ .

The propagator (3.6) then becomes

$$a(\vec{C}_{m-1}, m-1 \rightarrow \vec{C}_m, m) = (\sqrt{q})^{m-1-\sum_l C_{ml}} (\sqrt{p})^{\sum_l C_{ml}} (-i)^{\sum_i (1-\delta_{C_{mi} C_{m-1i}})}. \quad (3.9)$$

Using (3.4) and substituing (3.5), one finds that the model would be unitary with the slicing  $\{\vec{C}_m; m = 1, 2, \dots\}$  if one chose

$$\sum_{\vec{C}_m} a(\vec{C}_{m-1}, m-1 \rightarrow \vec{C}_m, m) a^*(\vec{C}_{m-1}, m-1 \rightarrow \vec{C}_m, m) = \prod_{l=1}^{m-1} \delta_{C_{m-1l} C'_{m-1l}},$$

i.e. if  $p = q = 1/2$ .

To simplify the notation, we will introduce the total number of entries equal to one in the  $m$ 'th column of the connectivity matrix,

$$C_m = \sum_{i=1}^N C_{im}, \quad (3.10)$$

and the number of “kinks” in each column,

$$K_m = \sum_{i=m+1}^{N-1} (1 - \delta_{C_{im} C_{i+1m}}). \quad (3.11)$$

The sum of these quantities over all of the columns of the connectivity matrix yield the total connectivity  $C$  and total kink number  $K$ , respectively.

We then arrive at a simple expression for the amplitude of a connectivity matrix (or “history”),

$$a(\mathbf{C}) = (\sqrt{p})^C (\sqrt{q})^{C_2^N - C} (-i)^K, \quad (3.12)$$

where  $C_2^N$  is the binomial coefficient.

The quantum measure of a set  $A$  of connectivity matrices is then given

$$|A| = \sum_{\vec{C}_N} |\psi(A, \vec{C}_N, N)|^2, \quad (3.13)$$

where

$$\psi(A, \vec{C}_N, N) \stackrel{\text{def}}{=} \sum_{\mathbf{C} \in A: \vec{C}_N \text{ fixed}} a(\mathbf{C}). \quad (3.14)$$

## 4 Analytical and Numerical Results

### 4.1 Measure of the space of histories, $|\mathcal{H}|$

As a first example one can compute the measure of the space  $\mathcal{H}$  of all possible non-cyclic oriented graphs of  $N$  points. In that case, one can drop the label  $A$  in (3.14) and write

$$|\mathcal{H}| = \sum_{\vec{C}_N} |\psi(\vec{C}_N, N)|^2. \quad (4.1)$$

The argument of the square modulus can be loosely interpreted as a “cosmological wave function”. To compute

$$\psi(\vec{C}_N, N) \stackrel{\text{def}}{=} \sum_{\mathbf{C}: \vec{C}_N \text{ fixed}} (\sqrt{p})^C (\sqrt{q})^{C_2^N - C} (-i)^K, \quad (4.2)$$

we use the fact that  $C = \sum_m C_m$  and  $K = \sum_m K_m$  to factorize the expression above and consider each column of the connectivity matrix individually:

$$\psi(\vec{C}_N, N) = (\sqrt{q})^{C_2^N} \prod_{m=1}^{N-1} \psi_m(C_{Nm}, N), \quad (4.3)$$

$$\psi_m(C_{Nm}, N) \stackrel{\text{def}}{=} \sum_{C_m=C_{Nm}}^{N-m+C_{Nm}-1} \sum_{K_m(C_m)} \left(\sqrt{\frac{p}{q}}\right)^{C_m} (-i)^{K_m} N_{C_{Nm}}(K_m, C_m), \quad (4.4)$$

where  $N_{C_{Nm}}(K_m, C_m)$  is the number of binary words of  $N - m - 1$  bits with  $C_m$  bits equal to 1 and  $K_m$  kinks, when the last bit in the column has been set equal to  $C_{Nm}$ .

The functions  $N_{C_{Nm}}(K_m, C_m)$  can be calculated by considering the number of ways of making  $k$  cuts in a sequence of  $C_m$  ones and inserting the zeroes at the cuts. One finds

$$N_0(2k, C_m) = \delta(k, 0)\delta(C_m, 0) + \binom{C_m - 1}{k - 1} \binom{N - m - 1 - C_m}{k}, \quad (4.5)$$

$$N_1(2k, C_m) = \delta(k, 0)\delta(C_m, N-m) + \binom{C_m-1}{k} \binom{N-m-1-C_m}{k-1}, \quad (4.6)$$

$$N_0(2k+1, C_m) = \binom{C_m-1}{k} \binom{N-m-1-C_m}{k}, \quad (4.7)$$

$$N_1(2k+1, C_m) = \binom{C_m-1}{k} \binom{N-m-1-C_m}{k}. \quad (4.8)$$

To determine the bounds on  $K_m$ , we must regard its dependence on  $C_{Nm}$  and make a comparison between the number of bits equal to 1,  $C_m$ , and the number of bits equal to zero,  $N-m-C_m$ , analyzing the different possible arrangements. The result is:

$$\text{if } N-m-C_m < C_m \leq N-m+C_{Nm}-1 \implies$$

$$K_m \in [1-C_{Nm}, 2(N-m-C_m+C_{Nm}-1)+1-C_{Nm}],$$

$$\text{if } N-m-C_m = C_m \implies K_m \in [1, 2C_m-1],$$

$$\text{if } C_{Nm} \leq C_m < N-m-C_m \implies K_m \in [C_{Nm}, 2C_m-C_{Nm}].$$

These equations allow one to calculate the function  $\psi(\vec{C}_N, N)$  on a computer and compute the measure  $|\mathcal{H}|$ . Not surprisingly, this measure is equal to 1 in the “unitary case”  $p = q = 1/2$  when  $\psi$  can be interpreted as a cosmological wave function. This is not particularly relevant in the context of the quantum measure interpretation.

#### 4.2 Graphs with a Fixed Number of Arrows from a Given Point

Setting aside for the time being the issue of labeling invariance, we will compute the propensity that a point with a given label emit a fixed number of arrows towards other points of the graph. Let  $A_{C_l}(l)$  be the set of histories (graphs) where point labeled  $l$  emits  $C_l$  outgoing arrows. From (3.12-3.14) we have

$$|A_{C_l}(l)| = \sum_{\vec{C}_N} |\psi(A_{C_l}(l), \vec{C}_N, N)|^2, \quad (4.9)$$



where

$$\psi(A_{C_l}(l), \vec{C}_N, N) = (\sqrt{q})^{C_l^N} \left(\sqrt{\frac{p}{q}}\right)^{C_{NN-1}} \psi_l(A_{C_l}(l), C_{Nl}, N) \prod_{\substack{m=1 \\ m \neq l}}^{N-2} \psi_m(C_{Nm}, N) \quad (4.10)$$

and

$$\psi_l(A_{C_l}(l), C_{Nl}, N) \stackrel{\text{def}}{=} \sum_{\vec{V}_l: A_{C_l}(l)} \left(\sqrt{\frac{p}{q}}\right)^{C_l} (-i)^{K_l}, \quad (4.11)$$

where we have used the short-hand notation  $\vec{V}_l: A_{C_l}(l)$  to denote the sum runs over the columns of binary words  $\vec{V}_l$ , which satisfy that the number of connections  $C_l$  be fixed. One finds

$$\frac{|A_{C_l}(l)|}{|\mathcal{H}|} = \left(\frac{p}{q}\right)^{C_l} \frac{|Z_0|^2 + |Z_1|^2}{|\psi_l(0, N)|^2 + |\psi_l(1, N)|^2}, \quad (4.12)$$

where

$$Z_{C_{Nl}} = \sum_{K_l} (-i)^{K_l} N_{C_{Nl}}(K_l, C_l) \quad (4.13)$$

represents the quantum interference factor. This ratio was computed numerically for  $p = q = 1/2$ , as a function of  $C_l$ .

In the classical directed percolation model, the probability of  $C_l$  is given by a binomial distribution

$$p(C_l) = \binom{N-l-1}{C_l} p^{C_l} (1-p)^{N-l-1-C_l}. \quad (4.14)$$

The ratio of the quantum measure to its classical counterpart,

$$F(l) = \frac{|A_{C_l}(l)|}{p(C_l)} \quad (4.15)$$

is a form factor which can be interpreted as representing the effect of quantum interference. A comparison between [Figure 1] and the binomial distribution reveals that the destructive interference is most important at the midpoint

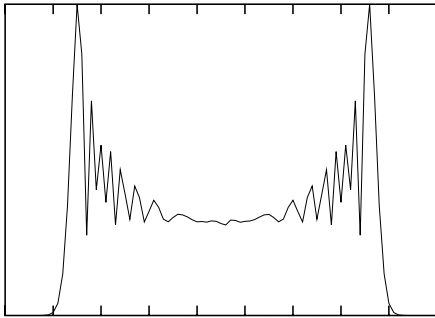


Figure 1: The propensity that a point (point number 90 from the final point) emits  $C$  arrows is represented as a function of  $C$ , for the case  $p = q = 1/2$ . The sharp rise and fall at both ends are remnants of a classical binomial distribution, whilst in the middle quantum interference is observed. Contrary to the classical case the propensity does not peak at  $C = 45$ , half the possible number of arrows.

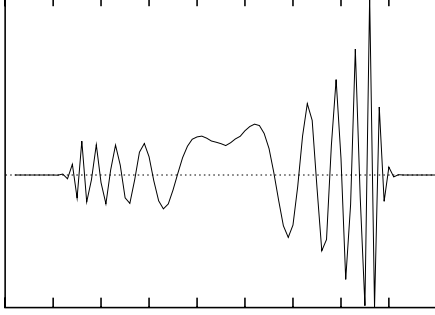


Figure 2: The real part of the form factor which gave rise to the previous figure is represented here.

where half the available bits are equal to one. When  $C_l$  is nearer one of the two extremal values one observes the same exponential dropoff as in the statistical model.

In the crossover between these two regimes, a striking interference pattern is observed. To witness the origin of this interference we show the real part of the form factor  $Z_0$  (the imaginary part is its mirror image) [Figure 2].

#### 4.3 Graphs with No Arrows from a Given Point

When the number of out-arrows from a point is equal to zero this point is a sink in the oriented graph. It is then not emitting any outgoing signals and therefore might loosely be called a classical “black hole”. The propensity for a point labeled  $l$  to be a sink is given by

$$\frac{|A_0(l)|}{|\bar{A}_0(l)|} = \frac{\sum_{\vec{C}_N} |\psi(A_0(l), \vec{C}_N, N)|^2}{\sum_{\vec{C}_N} |\psi(\bar{A}_0(l), \vec{C}_N, N)|^2}, \quad (4.16)$$

where  $\bar{A}_0(l)$  is the complement of the set  $A_0(l)$ .

With  $A_0(l) = \{\mathbf{C} : \vec{V}_l = (0, 0, \dots, C_{Nl}), l \notin \{1, N-1, N\}\}$ , we have

$$\psi(\bar{A}_0(l), \vec{C}_N, N) = \psi(\vec{C}_N, N) - \psi(A_0(l), \vec{C}_N, N), \quad (4.17)$$

$$\frac{|A_0(l)|}{|\bar{A}_0(l)|} = \frac{1 + \frac{p}{q}}{|\psi_l(0, N) - 1|^2 + |\psi_l(1, N) + i\sqrt{\frac{p}{q}}|^2}. \quad (4.18)$$

The sink propensity is represented as a function of the parameter  $p$  in [Figure 3], for  $N - l = 100$  and  $200$ . On the same figure we represented the corresponding classical expression  $q^{N-l}/(1 - q^{N-l})$  for comparison. Not surprisingly the classical curve crosses the line  $y = 1$  at a value of  $p$  that is just a bit lower than the percolation point  $1/N$ . The ratio of the quantum measures in contrast remains close to this line for a range of values of  $p$  covering three orders of magnitude. The peak in [Figure 3b] indicates a high sink propensity at  $N - l = 200$  for  $p \approx 0.036$ . This peak shows that an effective inhomogeneity is induced from the dependence of the measure on the labeling of the points. This labeling introduces the order of the natural numbers. Natural ordering enters in two different places: in the definition of a “kink”, and in the notion of a “sink”, which treats out-arrows differently from in-arrows and thereby gives relevance to the “distance”  $N - l$ . The existence of the points between  $N$  and  $l$  is not relevant for the causal order.

To analyze this effective inhomogeneity, the code was modified to scan the  $p$ -axis for a possible peak, for every value of  $N - l$ . This effort was distributed over several computers to arrive at the results represented in [Figure 4]. Very high peaks are observed for particular values of  $N - l$ , signaling points that almost certainly will be sinks.

#### 4.4 The Measure of Spacetimes with No Black Holes

An example of a labeling invariant question is “does spacetime have no

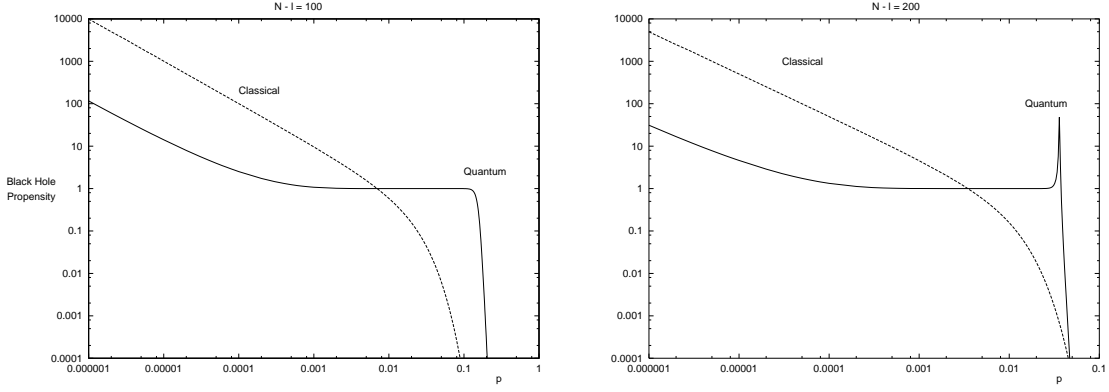


Figure 3: For lower values of  $p$  the classical and quantum theories behave differently: in classical percolation one has a phase transition at  $p_c \sim 1/N$  where the propensity that an arbitrary point is a sink is about equal to the propensity that it is not a sink. This is shown in the figure as a dotted line which represents the ratio of these two propensities. In the quantum model this ratio remains equal to one over an extended range of values of  $p$ . For some points such as  $N - l = 200$  (Fig. 3b), a peak in the black hole propensity ratio can occur at large values of  $p$ , contrary to classical intuition.

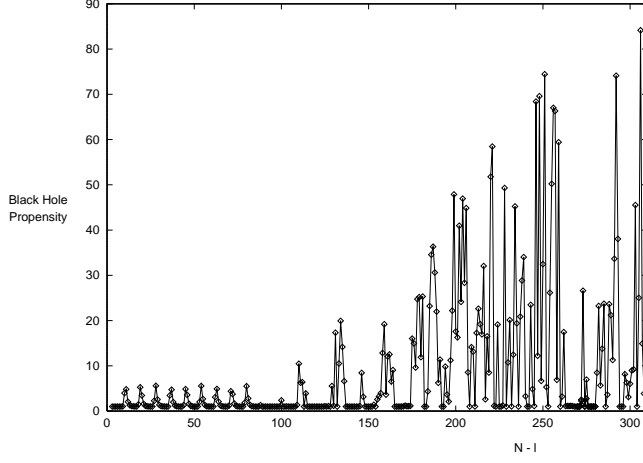


Figure 4: When a peak in the black hole propensity ratio is observed, the peak value of the propensity ratio is given in this graph. Different points here are obtained with different values of  $p$ , for each point the peak is located first and only the highest point is represented. Some points like  $N - l = 306$  can produce a significantly large ratio, indicating the likely occurrence of a sink for a particular value of the parameter  $p$ .

black holes?”. Let  $\bar{B}_0$  denote the set of graphs with no sinks and its complement  $B_0$  the set of graphs with one or more sinks. Then,

$$\psi(\bar{B}_0, \vec{C}_N, N) = (\sqrt{q})^{C_N^2} \prod_{m=1}^{N-1} \sum_{C_m, K_m} \left( \sqrt{\frac{p}{q}} \right)^{C_m} (-i)^{K_m} N_{C_{Nm}}(K_m, C_m) (1 - \delta(C_m - C_{Nm})) \quad (4.19)$$

and

$$\psi(B_0, \vec{C}_N, N) = \psi(\vec{C}_N, N) - \psi(\bar{B}_0, \vec{C}_N, N). \quad (4.20)$$

Using the above equations one finds

$$\frac{|\bar{B}_0|}{|B_0|} = \frac{\prod_m \left( |\psi_m(0) - 1|^2 + |\psi_m(1) + i\sqrt{\frac{p}{q}}|^2 \right)}{\prod_m (|\psi_m(0)|^2 + |\psi_m(1)|^2) - 2\text{Re}(W) + \prod_m \left( |\psi_m(0) - 1|^2 + |\psi_m(1) + i\sqrt{\frac{p}{q}}|^2 \right)}, \quad (4.21)$$

where

$$W = \prod_{m=1}^{N-2} \left( |\psi_m(0)|^2 - \psi_m(0) + |\psi_m(1)|^2 - i\sqrt{\frac{p}{q}}\psi_m(1) \right). \quad (4.22)$$

The sink propensity is very large for small values of  $p$ , since connections in that case are rare for statistical reasons, and vice versa it will drop to zero as the connection probability becomes large. A numerical evaluation of (4.21-4.22) shows however that the quantum model once again displays a broad region where the ratio is equal to one and the model simply does not provide any information as to whether or not one should expect to find any sinks [Figure 5]. A peak in the sink propensity for values of  $p$  well above the classical percolation point at  $p = 1/N$  reflects the presence of quantum interference effects.

## 5 Discussion

We presented a general procedure to construct a quantum model in the quantum measure formalism, and applied this procedure to arrive at a toy model

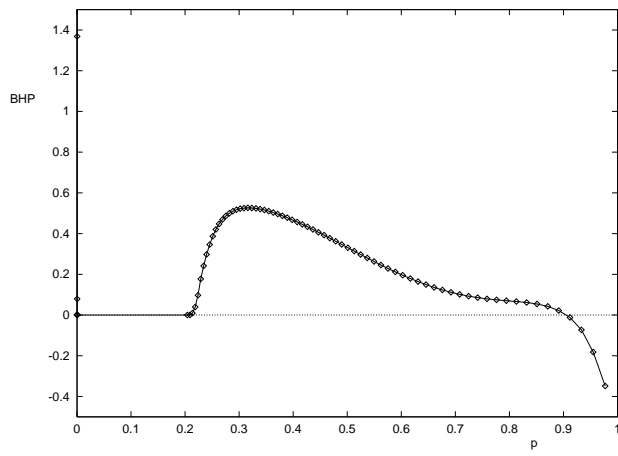


Figure 5: The “black hole propensity” is represented as a function of the connectivity parameter  $p$ .



of causal set dynamics.

Technical problems related to the transitivity condition on causal relations were circumvented by considering *non-cyclic oriented graphs* in place of posets. In a graph, one distinguishes transitive relations that are represented with an arrow to those that are not, even though this distinction is not relevant in terms of the causal order. Yet, physically relevant information can be derived from this model, by considering a limited class of physical properties that condition the connectivity matrix without raising the computational complexity to the same extent as the transitivity condition on a relations matrix (3.1).

An alternate strategy would be to regard the information in representing a transitive connection as an arrow to be physically significant and related to the propagation of a signal between two points. One would then be dealing with a larger theory, which describes gravity together with some other discrete excitations with a role similar to vector bosons.

This model may be relevant to other areas than causal sets and quantum gravity, if viewed as a quantum mechanical model for percolation. In this light, it is well worth recalling the observation that the quantum model displays two distinct transitions, from an “underconnected” phase where almost all points are isolated, through a broad “quantum phase”, and finally to a “connected” phase where almost all points belong to the same dominant cluster. This contrasts with a classical percolation model where the transition between the first and last phases occurs at a critical point. A similar situation is often found in statistical models such as traffic models and other self-organized critical systems. In a traffic model for example, the forcing parameter is the rate at which vehicles are placed at one end of the system looking for an opportunity to proceed. With low forcing one finds a constant mean vehicle speed limited only by the circulating velocity and the presence of traffic lights. At intermediate values of the forcing parameter the entry point can become obstructed and this limits further increases in the flow of

traffic. This phase is characterized by a scale-invariant behaviour of the rate of flow. At yet higher forcing levels, a second phase transition takes place, to a completely saturated situation with an exponential cutoff in velocity fluctuations. The possibility that the quantum generalization of a directed percolation model produce phenomena similar to those that are generically found in self-organized critical phenomena is of course tantalizing, particularly if this can help avoid the need to fine-tune parameters of the system to match theory with experiment.

The measure calculation of section (4.2) hints that it may be possible to eliminate the need for fine-tuning a theory at the critical point without breaking Lorentz invariance. The extended plateau in sink propensity shows that there is a large range of values of  $p$  where the quantum model behaves neither like a classical model below the critical point nor like a classical model above the critical point. We do not know whether this plateau indicates the presence of a different “quantum percolation phase”, with properties unlike those of any classical stochastic model, or the extension of a classical critical regime characterized by a power-law distribution of cluster sizes, as in self-organized critical systems.

To make statements about percolation more precise one would have to consider the measure of the set of histories such that two given points  $k, l$  are related, either by a direct connection or by a transitive chain of connections. Unfortunately this immediately leads one into the difficulty that we tried to avoid by working with the connection matrix rather than the relations matrix: the transitivity condition requires that one consider powers of the matrix  $\mathbf{C}$  up to an order that increases linearly with  $N$ , only to determine whether or not there exists a transitive relation between two points. This implies that the measure cannot be computed using factorization, as in section 4, and calculation by exhaustive summation is limited to about  $N = 11$  with a modern supercomputer.

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## 6 Figure captions

Figure 1. The propensity that a point (point number 90 from the final point) emits  $C$  arrows is represented as a function of  $C$ , for the case  $p = q = 1/2$ . The sharp rise and fall at both ends are remnants of a classical binomial distribution, whilst in the middle quantum interference is observed. Contrary to the classical case the propensity does not peak at  $C = 45$ , half the possible number of arrows.

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